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## LORENTZIAN STATIONARY SURFACES IN 4-DIMENSIONAL SPACE FORMS OF INDEX 2

By

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**Abstract.** We discuss the necessary and sufficient conditions for the existence of Lorentzian stationary surfaces in 4-dimensional space forms of index 2, and isometric stationary deformations preserving normal curvature.

### 1. Introduction

Let  $N_p^n(c)$  denote the  $n$ -dimensional semi-Riemannian space form of constant curvature  $c$  and index  $p$ . Namely, it is the  $n$ -dimensional semi-Euclidean space  $R_p^n$  of index  $p$ , the  $n$ -dimensional pseudo-sphere  $S_p^n(c)$  of constant curvature  $c > 0$  and index  $p$ , or the  $n$ -dimensional pseudo-hyperbolic space  $H_p^n(c)$  of constant curvature  $c < 0$  and index  $p$ . We write  $N^n(c)$  if  $p = 0$ . A surface in  $N_p^n(c)$  is called Lorentzian if the induced metric is Lorentzian. We shall say that a Lorentzian surface in  $N_p^n(c)$  is stationary if the mean curvature vector is identically zero.

For a minimal surface in  $N^4(c)$ , the Gaussian curvature  $K(\leq c)$  and the normal curvature  $K_\nu$  satisfy  $(K - c)^2 - K_\nu^2 \geq 0$ , where the equality holds at isotropic points. In [12] Tribuzy and Guadalupe give the necessary and sufficient conditions for the existence of minimal surfaces in  $N^4(c)$  in terms of the metric and the normal curvature, and discuss isometric minimal deformations preserving normal curvature. Also, for a spacelike maximal surface in  $N_2^4(c)$ ,  $K(\geq c)$  and  $K_\nu$  satisfy  $(K - c)^2 - K_\nu^2 \geq 0$ , where the equality holds at isotropic points. In a previous paper [7], we give the necessary and sufficient conditions for the existence of spacelike maximal surfaces in  $N_2^4(c)$ , and discuss isometric maximal deformations preserving normal curvature.

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For a Lorentzian stationary surface in  $N_2^4(c)$ , the signs of  $K - c$  and  $(K - c)^2 - K_v^2$  are not fixed, and it seems that there are many different situations compared with the case of minimal surfaces in  $N^4(c)$  or spacelike maximal surfaces in  $N_2^4(c)$ . In this paper, we will discuss the necessary and sufficient conditions for the existence of Lorentzian stationary surfaces in  $N_2^4(c)$ , and isometric stationary deformations preserving normal curvature.

The results are stated as follows:

**THEOREM 1.1.** (i) *Let  $M$  be a Lorentzian stationary surface in  $N_2^4(c)$  with Gaussian curvature  $K$ , normal curvature  $K_v$  and Laplacian  $\Delta$ . If  $(K - c)^2 - K_v^2 \neq 0$ , then*

$$\Delta \log|K - c + K_v| = 2(2K + K_v), \quad (1)$$

$$\Delta \log|K - c - K_v| = 2(2K - K_v). \quad (2)$$

(ii) *Let  $M$  be a 2-dimensional simply connected Lorentzian manifold with Gaussian curvature  $K$  and Laplacian  $\Delta$ . If  $K_v$  is a function on  $M$  satisfying  $(K - c)^2 - K_v^2 > 0$  and (1), (2), then there exists an isometric stationary immersion of  $M$  into  $N_2^4(c)$  with normal curvature  $K_v$ .*

**THEOREM 1.2.** *Let  $f : M \rightarrow N_2^4(c)$  be an isometric stationary immersion of a 2-dimensional simply connected Lorentzian manifold  $M$  into  $N_2^4(c)$  with Gaussian curvature  $K$  and normal curvature  $K_v$ .*

(i) *There exist two one-parameter families of isometric stationary immersions  $f_\theta, \tilde{f}_\theta : M \rightarrow N_2^4(c)$  ( $\theta \in \mathbb{R}$ ) with the same normal curvature  $K_v$ .*

(ii) *Assume that  $(K - c)^2 - K_v^2 > 0$ . If  $\hat{f} : M \rightarrow N_2^4(c)$  is an arbitrary isometric stationary immersion with the same normal curvature  $K_v$ , then there exists  $\theta \in \mathbb{R}$  such that  $\hat{f}$  coincides with  $f_\theta$  or  $\tilde{f}_\theta$  up to congruence.*

**REMARK.** The theorems and their proof imply that Lorentzian stationary surfaces in  $N_1^3(c)$  and  $N_2^3(c)$  with  $K \neq c$  are intrinsically characterized by the same condition  $\Delta \log|K - c| = 4K$ , and each of them has a one-parameter family of isometric stationary immersions into  $N_1^3(c)$  and  $N_2^3(c)$ , respectively. So, viewing  $N_1^3(c)$  and  $N_2^3(c)$  as subspaces of  $N_2^4(c)$ , a Lorentzian stationary surface in  $N_1^3(c)$  or  $N_2^3(c)$  with  $K \neq c$  has two one-parameter families of isometric stationary immersions with zero normal curvature into  $N_2^4(c)$ . Theorem 1.2 is a natural generalization of this situation.

The theorems say that, in the case where  $(K - c)^2 - K_v^2 > 0$ , Lorentzian stationary surfaces in  $N_2^4(c)$  have similar properties to minimal surfaces in  $N^4(c)$

or spacelike maximal surfaces in  $N_2^4(c)$  (cf. [12] and [7]), except for the existence of two kinds of isometric stationary deformations preserving normal curvature. But, in the case where  $(K - c)^2 - K_v^2 \leq 0$ , we do not know how is a Lorentzian stationary surface in  $N_2^4(c)$  determined by the metric and the normal curvature. As will be noted in the last section, the crucial different point is that a certain symmetric linear transformation of the normal bundle can be diagonalized or not.

The study of Lorentzian stationary surfaces in  $N_2^4(c)$  may be seen as a special case of that of real parakähler submanifolds in  $N_p^n(c)$ , namely, isometric immersions of parakähler manifolds into  $N_p^n(c)$ , in particular, in the case of zero mean curvature. The results in this paper suggest that the geometry of real parakähler submanifolds may be quite different from that of real Kähler submanifolds (cf. [4], [3], [2] and their references).

## 2. Preliminaries

In this section, we recall the method of moving frames for Lorentzian surfaces in  $N_2^4(c)$ . Unless otherwise stated, we use the following convention on the ranges of indices:

$$1 \leq A, B, \dots \leq 4, \quad 1 \leq i, j, \dots \leq 2, \quad 3 \leq \alpha, \beta, \dots \leq 4.$$

Let  $\{e_A\}$  be a local orthonormal frame field in  $N_2^4(c)$ , and  $\{\omega^A\}$  be the dual coframe field, so that the metric of  $N_2^4(c)$  is given by

$$d\sigma^2 = (\omega^1)^2 - (\omega^2)^2 + (\omega^3)^2 - (\omega^4)^2.$$

The connection forms  $\{\omega_B^A\}$  are defined by

$$de_B = \sum_A \omega_B^A e_A.$$

Then,  $\omega_B^A = -\omega_A^B$  if  $|A - B|$  is even, and  $\omega_B^A = \omega_A^B$  if  $|A - B|$  is odd. The structure equations are given by

$$d\omega^A = - \sum_B \omega_B^A \wedge \omega^B, \quad (3)$$

$$d\omega_B^A = - \sum_C \omega_C^A \wedge \omega_B^C + \frac{1}{2} \sum_{C,D} R_{BCD}^A \omega^C \wedge \omega^D, \quad (4)$$

$$R_{BCD}^A = c\varepsilon_B(\delta_C^A \delta_{BD} - \delta_D^A \delta_{BC}), \quad (5)$$

where  $\varepsilon_1 = \varepsilon_3 = 1$  and  $\varepsilon_2 = \varepsilon_4 = -1$ .

Let  $M$  be a Lorentzian surface in  $N_2^4(c)$ . We choose the frame  $\{e_A\}$  so that  $\{e_i\}$  are tangent to  $M$ . Then  $\omega^\alpha = 0$  on  $M$ . In the following our argument will be restricted to  $M$ . By (3),

$$0 = - \sum_i \omega_i^\alpha \wedge \omega^i.$$

So there is a symmetric tensor  $\{h_{ij}^\alpha\}$  such that

$$\omega_i^\alpha = \sum_j h_{ij}^\alpha \omega^j, \quad (6)$$

where  $h_{ij}^\alpha$  are the components of the second fundamental form  $h$  of  $M$ .

The Gaussian curvature  $K$  and the normal curvature  $K_v$  of  $M$  are given by

$$d\omega_2^1 = -K\omega^1 \wedge \omega^2, \quad d\omega_4^3 = -K_v\omega^1 \wedge \omega^2. \quad (7)$$

Then by (4), (5) and (6), we have

$$K = c - h_{11}^3 h_{22}^3 + (h_{12}^3)^2 + h_{11}^4 h_{22}^4 - (h_{12}^4)^2, \quad (8)$$

and

$$K_v = h_{11}^3 h_{12}^4 - h_{12}^3 h_{11}^4 - h_{12}^3 h_{22}^4 + h_{22}^3 h_{12}^4. \quad (9)$$

The mean curvature vector  $H$  of  $M$  is defined by

$$H = \frac{1}{2} \sum_\alpha (h_{11}^\alpha - h_{22}^\alpha) e_\alpha.$$

We say that  $M$  is stationary if  $H = 0$  on  $M$ .

In the following we assume that  $M$  is stationary. Then by (8) and (9),

$$K = c - (h_{11}^3)^2 + (h_{12}^3)^2 + (h_{11}^4)^2 - (h_{12}^4)^2, \quad (10)$$

and

$$K_v = 2(h_{11}^3 h_{12}^4 - h_{12}^3 h_{11}^4). \quad (11)$$

We can see that

$$\begin{aligned} (K - c)^2 - K_v^2 &= \{-(h_{11}^3)^2 + (h_{12}^3)^2 + (h_{11}^4)^2 - (h_{12}^4)^2\}^2 - 4(h_{11}^3 h_{12}^4 - h_{12}^3 h_{11}^4)^2 \\ &= \{(h_{11}^3)^2 + (h_{12}^3)^2 - (h_{11}^4)^2 - (h_{12}^4)^2\}^2 - 4(h_{11}^3 h_{12}^3 - h_{11}^4 h_{12}^4)^2 \\ &= \{(h_{11}^3)^2 - (h_{12}^3)^2 + (h_{11}^4)^2 - (h_{12}^4)^2\}^2 - 4(h_{11}^3 h_{11}^4 - h_{12}^3 h_{12}^4)^2. \end{aligned} \quad (12)$$

### 3. Some Examples

In this section, we give some examples of Lorentzian stationary surfaces in  $N_2^4(c)$ .

EXAMPLE 3.1. Let  $\{x_1, x_2, x_3, x_4\}$  be the standard coordinate system for  $R_2^4$  with metric

$$d\sigma^2 = dx_1^2 - dx_2^2 + dx_3^2 - dx_4^2.$$

Let  $\bar{J}$  be the paracomplex structure on  $R_2^4$  given by

$$\bar{J}\left(\frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial x_2}, \quad \bar{J}\left(\frac{\partial}{\partial x_2}\right) = \frac{\partial}{\partial x_1}, \quad \bar{J}\left(\frac{\partial}{\partial x_3}\right) = \frac{\partial}{\partial x_4}, \quad \bar{J}\left(\frac{\partial}{\partial x_4}\right) = \frac{\partial}{\partial x_3}.$$

Then  $(\bar{J}, d\sigma^2)$  is a flat parakähler structure on  $R_2^4$ .

Let  $M$  be a paracomplex surface in  $R_2^4$ , that is,  $\bar{J}(T_p M) = T_p M$  for each  $p \in M$ . Then, by Corollary 3.1 of [1],  $M$  is a Lorentzian stationary surface in  $R_2^4$ . For example, set

$$f(u, v) = \begin{pmatrix} Q_1(u+v) + Q_2(u-v) \\ Q_1(u+v) - Q_2(u-v) \\ Q_3(u+v) + Q_4(u-v) \\ Q_3(u+v) - Q_4(u-v) \end{pmatrix},$$

and assume that

$$Q'_1(u+v)Q'_2(u-v) + Q'_3(u+v)Q'_4(u-v) > 0,$$

where  $Q_1(z)$ ,  $Q_2(z)$ ,  $Q_3(z)$  and  $Q_4(z)$  are smooth functions. Then it gives a paracomplex surface in  $R_2^4$ . See [9] and [11] for a relation between paracomplex surfaces and minimal lightlike submanifolds.

EXAMPLE 3.2. For a constant  $k > 0$  and a smooth function  $Q(u)$  with  $Q'(u) > 0$ , let  $M$  be a surface in  $R_2^4$  given by

$$f(u, v) = (Q(u) \cosh v, Q(u) \sinh v, u, kv),$$

where  $R_2^4$  has the same metric as in Example 3.1. It is a deformation in  $R_2^4$  of a Lorentzian surface of revolution in  $R_1^3$  with spacelike axis of revolution (cf. [14, p. 350], [5, p. 520]). Set

$$e_1 = \frac{1}{\sqrt{1+Q'^2}} f_u = \frac{1}{\sqrt{1+Q'^2}} (Q' \cosh v, Q' \sinh v, 1, 0),$$

$$\begin{aligned}
e_2 &= \frac{1}{\sqrt{Q^2 + k^2}} f_v = \frac{1}{\sqrt{Q^2 + k^2}} (Q \sinh v, Q \cosh v, 0, k), \\
e_3 &= \frac{1}{\sqrt{1 + Q'^2}} (\cosh v, \sinh v, -Q', 0), \\
e_4 &= \frac{1}{\sqrt{Q^2 + k^2}} (k \sinh v, k \cosh v, 0, -Q).
\end{aligned}$$

Then  $\{e_A\}$  is an orthonormal frame field along  $M$  with signature  $(+, -, +, -)$ , and  $\{e_i\}$  are tangent to  $M$ . The components of the second fundamental form  $h$  are given by

$$\begin{aligned}
h_{11}^3 &= \frac{Q''}{(1 + Q'^2)^{3/2}}, \quad h_{12}^3 = 0, \quad h_{22}^3 = \frac{Q}{(Q^2 + k^2)\sqrt{1 + Q'^2}}, \\
h_{11}^4 &= 0, \quad h_{12}^4 = \frac{kQ'}{(Q^2 + k^2)\sqrt{1 + Q'^2}}, \quad h_{22}^4 = 0.
\end{aligned}$$

Thus  $M$  is stationary if and only if

$$\frac{Q''}{1 + Q'^2} = \frac{Q}{Q^2 + k^2}.$$

Multiplying by  $2Q'$  and integrating, we may obtain

$$Q' = \sqrt{c_1^2 Q^2 + c_1^2 k^2 - 1}, \quad (c_1 > 0).$$

If  $c_1 k = 1$ , then  $Q(u) = c_2 e^{u/k}$ . If  $c_1 k < 1$ , then

$$Q(u) = \frac{\sqrt{1 - c_1^2 k^2}}{c_1} \cosh(c_1 u + c_2).$$

If  $c_1 k > 1$ , then

$$Q(u) = \frac{\sqrt{c_1^2 k^2 - 1}}{c_1} \sinh(c_1 u + c_2).$$

EXAMPLE 3.3. Let  $M$  be a 2-dimensional simply connected Lorentzian manifold with Gaussian curvature  $K$  and Laplacian  $\Delta$ . Suppose that  $K > c$  and

$$\Delta \log(K - c) = 6K - 2c.$$

Then by Theorem 1 of [8], there exists an isometric stationary immersion of  $M$  into  $N_2^4(c)$ . This is an isotropic-like example.

EXAMPLE 3.4. Let  $P(u)$ ,  $Q(v)$  be null curves in  $R_2^4$ , and assume that  $\langle P'(u), Q'(v) \rangle \neq 0$ . Set  $f(u, v) = P(u) + Q(v)$ . Then it gives a Lorentzian stationary surface in  $R_2^4$ . See [13, Chap. 8] for such a representation in  $R_1^3$ . See [10] for the geometry of null curves in  $R_2^4$ .

#### 4. Proof of Theorem 1.1

PROOF OF THEOREM 1.1. (i) As  $M$  is a Lorentzian stationary surface in  $N_2^4(c)$ , using the notations in Section 2, we may write

$$\omega_1^3 = s\omega^1 + t\omega^2, \quad \omega_2^3 = t\omega^1 + s\omega^2, \quad \omega_1^4 = u\omega^1 + v\omega^2, \quad \omega_2^4 = v\omega^1 + u\omega^2. \quad (13)$$

By (10) and (11),

$$K = c - s^2 + t^2 + u^2 - v^2, \quad K_v = 2(sv - tu). \quad (14)$$

Using (3), (4), (5) and (13), we have

$$\begin{aligned} d\omega_1^3 &= ds \wedge \omega^1 - s\omega_2^1 \wedge \omega^2 + dt \wedge \omega^2 - t\omega_1^2 \wedge \omega^1 \\ &= -\omega_2^3 \wedge \omega_1^2 - \omega_4^3 \wedge \omega_1^4 \\ &= -(t\omega^1 + s\omega^2) \wedge \omega_1^2 - \omega_4^3 \wedge (u\omega^1 + v\omega^2). \end{aligned}$$

Using the notation like

$$\begin{aligned} ds &= s_1\omega^1 + s_2\omega^2, \quad dt = t_1\omega^1 + t_2\omega^2, \\ \omega_2^1 &= (\omega_2^1)_1\omega^1 + (\omega_2^1)_2\omega^2, \quad \omega_4^3 = (\omega_4^3)_1\omega^1 + (\omega_4^3)_2\omega^2, \end{aligned}$$

we get

$$2s(\omega_2^1)_1 - 2t(\omega_2^1)_2 - v(\omega_4^3)_1 + u(\omega_4^3)_2 = t_1 - s_2. \quad (15)$$

Similarly, from the exterior derivatives of  $\omega_2^3$ ,  $\omega_1^4$  and  $\omega_2^4$ ,

$$2s(\omega_2^1)_2 - 2t(\omega_2^1)_1 - v(\omega_4^3)_2 + u(\omega_4^3)_1 = t_2 - s_1, \quad (16)$$

$$2u(\omega_2^1)_1 - 2v(\omega_2^1)_2 - t(\omega_4^3)_1 + s(\omega_4^3)_2 = v_1 - u_2, \quad (17)$$

$$2u(\omega_2^1)_2 - 2v(\omega_2^1)_1 - t(\omega_4^3)_2 + s(\omega_4^3)_1 = v_2 - u_1. \quad (18)$$

Using (14) we can see that

$$\begin{vmatrix} s & -t & -v & u \\ -t & s & u & -v \\ u & -v & -t & s \\ -v & u & s & -t \end{vmatrix} = -\{(K - c)^2 - K_v^2\} \neq 0.$$



So the simultaneous linear equations (15)–(18) for  $2(\omega_2^1)_1$ ,  $2(\omega_2^1)_2$ ,  $(\omega_4^3)_1$  and  $(\omega_4^3)_2$  can be solved uniquely. From (15)–(18) we have

$$2s\omega_2^1 - 2t(*\omega_2^1) - v\omega_4^3 + u(*\omega_4^3) = dt - (*ds), \quad (19)$$

$$-2t\omega_2^1 + 2s(*\omega_2^1) + u\omega_4^3 - v(*\omega_4^3) = (*dt) - ds, \quad (20)$$

$$2u\omega_2^1 - 2v(*\omega_2^1) - t\omega_4^3 + s(*\omega_4^3) = dv - (*du), \quad (21)$$

$$-2v\omega_2^1 + 2u(*\omega_2^1) + s\omega_4^3 - t(*\omega_4^3) = (*dv) - du. \quad (22)$$

Here  $*$  is the Lorentzian Hodge star operator on  $M$  given by  $*\omega^1 = \omega^2$  and  $*\omega^2 = \omega^1$ .

By  $(19) \times (-s) + (20) \times (-t) + (21) \times u + (22) \times v$  and  $(19) \times v + (20) \times u + (21) \times (-t) + (22) \times (-s)$ , together with (14), we can get

$$2(K - c)\omega_2^1 + K_v\omega_4^3 = -\frac{1}{2}(*d(K - c)) + t\,ds - s\,dt - v\,du + u\,dv, \quad (23)$$

and

$$2K_v\omega_2^1 + (K - c)\omega_4^3 = -\frac{1}{2}(*dK_v) - u\,ds + v\,dt + s\,du - t\,dv. \quad (24)$$

Set

$$X = s^2 + t^2 - u^2 - v^2, \quad Y = 2(st - uv),$$

$$Z = s^2 - t^2 + u^2 - v^2, \quad W = 2(su - tv).$$

By (12) we have

$$(K - c)^2 - K_v^2 = X^2 - Y^2 = Z^2 - W^2. \quad (25)$$

Using (14) we can compute that

$$\begin{aligned} & (K - c)(t\,ds - s\,dt - v\,du + u\,dv) - K_v(-u\,ds + v\,dt + s\,du - t\,dv) \\ &= \frac{1}{2}(X\,dY - Y\,dX), \end{aligned} \quad (26)$$

and

$$\begin{aligned} & -K_v(t\,ds - s\,dt - v\,du + u\,dv) + (K - c)(-u\,ds + v\,dt + s\,du - t\,dv) \\ &= -\frac{1}{2}(Z\,dW - W\,dZ). \end{aligned} \quad (27)$$

By (23), (24), (25), (26) and (27), we can get

$$2\omega_2^1 = -\frac{1}{4} * d \log |(K-c)^2 - K_v^2| + \frac{1}{2} \cdot \frac{X dY - Y dX}{X^2 - Y^2}, \quad (28)$$

and

$$\omega_4^3 = \frac{1}{4} * d \log \left| \frac{K-c-K_v}{K-c+K_v} \right| - \frac{1}{2} \cdot \frac{Z dW - W dZ}{Z^2 - W^2}. \quad (29)$$

The Laplacian  $\Delta$  on  $M$  is given by

$$d * df = (\Delta f) \omega^1 \wedge \omega^2$$

for a smooth function  $f$  on  $M$ . By the exterior derivative of (28) and (29), together with (7), we may obtain

$$\Delta \log |(K-c)^2 - K_v^2| = 8K, \quad (30)$$

and

$$\Delta \log \left| \frac{K-c-K_v}{K-c+K_v} \right| = -4K_v. \quad (31)$$

By (30)±(31), we have the equations (1) and (2).

(ii) As  $(K-c)^2 - K_v^2 > 0$ , we have  $K \neq c$ . By the anti-isometry from  $N_2^4(c)$  to  $N_2^4(-c)$  (cf. [6, p. 110]),  $M$  remains Lorentzian, the Gaussian curvature, the normal curvature and the Laplacian turn to  $-K$ ,  $-K_v$  and  $-\Delta$ , respectively. So it suffices to consider only the case where  $K > c$ .

We may assume that  $M$  is a small neighborhood. Let  $d\sigma^2$  be the induced metric on  $M$ . By (1)+(2) we have

$$\Delta \log \{(K-c)^2 - K_v^2\} = 8K,$$

which implies that the metric

$$d\tilde{\sigma}^2 = \{(K-c)^2 - K_v^2\}^{1/4} d\sigma^2$$

is flat. So there exists a coordinate system  $\{x^1, x^2\}$  such that

$$d\sigma^2 = \{(K-c)^2 - K_v^2\}^{-1/4} \{(dx^1)^2 - (dx^2)^2\}.$$

Set

$$\omega^i = \{(K-c)^2 - K_v^2\}^{-1/8} dx^i, \quad (32)$$

so that  $\{\omega^i\}$  is an orthonormal coframe field dual to  $\{e_i\}$ . By

$$d\omega^1 = -\omega_2^1 \wedge \omega^2, \quad d\omega^2 = -\omega_1^2 \wedge \omega^1,$$

we can find that the connection form  $\omega_2^1 = \omega_1^2$  is given by

$$\omega_2^1 = \omega_1^2 = -\frac{1}{8} * d \log\{(K-c)^2 - K_v^2\}.$$

As  $(K-c)^2 - K_v^2 > 0$  and  $K > c$ , there exist smooth functions  $t$  and  $u$  so that

$$t^2 + u^2 = K - c, \quad tu = -\frac{1}{2}K_v.$$

In fact, letting

$$q = \frac{\sqrt{K-c-K_v} + \sqrt{K-c+K_v}}{2}, \quad (33)$$

and

$$r = \frac{\sqrt{K-c-K_v} - \sqrt{K-c+K_v}}{2}, \quad (34)$$

we have  $(t, u) = \pm(q, r)$ , or  $(t, u) = \pm(r, q)$ .

Let  $E$  be a 2-plane bundle over  $M$  with metric  $\langle, \rangle$  and orthonormal sections  $\{e_3, e_4\}$  of signature  $(+, -)$ . Let  $h$  be a symmetric section of  $\text{Hom}(TM \times TM, E)$  such that

$$(h_{ij}^3) = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix},$$

and set

$$\omega_1^3 = -\omega_3^1 = t\omega^2, \quad \omega_2^3 = \omega_3^2 = t\omega^1, \quad \omega_1^4 = \omega_4^1 = u\omega^1, \quad \omega_2^4 = -\omega_4^2 = u\omega^2.$$

We define a compatible connection  ${}^\perp\nabla$  of  $E$  so that

$${}^\perp\nabla e_3 = \omega_3^4 e_4, \quad {}^\perp\nabla e_4 = \omega_4^3 e_3,$$

where

$$\omega_4^3 = \omega_3^4 = \frac{1}{4} * d \log \left( \frac{K-c-K_v}{K-c+K_v} \right).$$

Now, almost reversing the argument in (i) for  $s = v = 0$ , we can see that  $\{\omega_B^A\}$  satisfy the structure equations:

$$\begin{aligned} d\omega_2^1 &= -\omega_3^1 \wedge \omega_2^3 - \omega_4^1 \wedge \omega_2^4 - c\omega^1 \wedge \omega^2, \\ d\omega_1^3 &= -\omega_2^3 \wedge \omega_1^2 - \omega_4^3 \wedge \omega_1^4, & d\omega_2^3 &= -\omega_1^3 \wedge \omega_2^1 - \omega_4^3 \wedge \omega_2^4, \\ d\omega_1^4 &= -\omega_2^4 \wedge \omega_1^2 - \omega_3^4 \wedge \omega_1^3, & d\omega_2^4 &= -\omega_1^4 \wedge \omega_2^1 - \omega_3^4 \wedge \omega_2^3, \\ d\omega_4^3 &= -\omega_1^3 \wedge \omega_4^1 - \omega_2^3 \wedge \omega_4^2, \end{aligned}$$

which are the integrability conditions. Hence, by the fundamental theorem, there exists an isometric immersion of  $M$  into  $N_2^4(c)$ , which is stationary and has normal curvature  $K_v$ .  $\square$

## 5. Proof of Theorem 1.2

PROOF OF THEOREM 1.2. (i) For  $f : M \rightarrow N_2^4(c)$ , let  $s, t, u, v$  and  $\omega_B^A$  be as in the proof of Theorem 1.1 (i). For each  $\theta \in R$ , let  $h(\theta)$  be a symmetric section of  $\text{Hom}(TM \times TM, T^\perp M)$  such that

$$\begin{aligned} (h_{ij}^3(\theta)) &= \begin{pmatrix} s \cosh 2\theta + t \sinh 2\theta & s \sinh 2\theta + t \cosh 2\theta \\ s \sinh 2\theta + t \cosh 2\theta & s \cosh 2\theta + t \sinh 2\theta \end{pmatrix}, \\ (h_{ij}^4(\theta)) &= \begin{pmatrix} u \cosh 2\theta + v \sinh 2\theta & u \sinh 2\theta + v \cosh 2\theta \\ u \sinh 2\theta + v \cosh 2\theta & u \cosh 2\theta + v \sinh 2\theta \end{pmatrix}, \end{aligned}$$

and correspondingly, set

$$\begin{aligned} \omega_1^3(\theta) &= -\omega_3^1(\theta) = \omega_1^3 \cosh 2\theta + \omega_2^3 \sinh 2\theta, \\ \omega_2^3(\theta) &= \omega_3^2(\theta) = \omega_1^3 \sinh 2\theta + \omega_2^3 \cosh 2\theta, \\ \omega_1^4(\theta) &= \omega_4^1(\theta) = \omega_1^4 \cosh 2\theta + \omega_2^4 \sinh 2\theta, \\ \omega_2^4(\theta) &= -\omega_4^2(\theta) = \omega_1^4 \sinh 2\theta + \omega_2^4 \cosh 2\theta. \end{aligned}$$

Let  $\omega_2^1(\theta) = \omega_1^2(\theta) = \omega_2^1$  and  $\omega_4^3(\theta) = \omega_3^4(\theta) = \omega_4^3$  for convenience. Then by the computation, we can see that  $\{\omega_B^A(\theta)\}$  satisfy the structure equations. Thus, for each  $\theta \in R$ , there exists an isometric immersion  $f_\theta : M \rightarrow N_2^4(c)$ , which is stationary and has the same normal curvature  $K_v$ .

Next, let  $\bar{h}$  be a symmetric section of  $\text{Hom}(TM \times TM, T^\perp M)$  such that

$$(\bar{h}_{ij}^3) = \begin{pmatrix} v & u \\ u & v \end{pmatrix}, \quad (\bar{h}_{ij}^4) = \begin{pmatrix} t & s \\ s & t \end{pmatrix},$$

and correspondingly, set

$$\begin{aligned}\bar{\omega}_1^3 &= -\bar{\omega}_3^1 = \omega_2^4, & \bar{\omega}_2^3 &= \bar{\omega}_3^2 = \omega_1^4, \\ \bar{\omega}_1^4 &= \bar{\omega}_4^1 = \omega_2^3, & \bar{\omega}_2^4 &= -\bar{\omega}_4^2 = \omega_1^3.\end{aligned}$$

Let  $\bar{\omega}_2^1 = \bar{\omega}_1^2 = \omega_2^1$  and  $\bar{\omega}_4^3 = \bar{\omega}_3^4 = \omega_4^3$  for convenience. By the computation, we can see that  $\{\bar{\omega}_B^A\}$  satisfy the structure equations. So there exists an isometric immersion  $\bar{f}: M \rightarrow N_2^4(c)$ , which is stationary and has the same normal curvature  $K_\nu$ .

Combining the above two methods, we get two one-parameter families of isometric stationary immersions  $f_\theta, \bar{f}_\theta: M \rightarrow N_2^4(c)$  ( $\theta \in \mathbb{R}$ ) with the same normal curvature  $K_\nu$ .  $\square$

Before proving the part (ii), we shall prepare a lemma. Let  $M$  be a Lorentzian stationary surface in  $N_2^4(c)$  satisfying  $(K - c)^2 - K_\nu^2 > 0$ . As in the proof of Theorem 1.1 (ii), we may assume that  $K > c$ . As  $(K - c)^2 - K_\nu^2 > 0$ , by (12), we may choose a smooth function  $\varphi$  so that

$$\{(h_{11}^3)^2 - (h_{12}^3)^2 + (h_{11}^4)^2 - (h_{12}^4)^2\} \sinh 2\varphi + 2(h_{11}^3 h_{11}^4 - h_{12}^3 h_{12}^4) \cosh 2\varphi = 0.$$

Set

$$\bar{e}_3 = e_3 \cosh \varphi + e_4 \sinh \varphi, \quad \bar{e}_4 = e_3 \sinh \varphi + e_4 \cosh \varphi,$$

and let  $\{\bar{h}_{ij}^\alpha\}$  be the components of  $h$  with respect to  $\{e_i, \bar{e}_\alpha\}$ . Then we may have

$$\bar{h}_{11}^3 \bar{h}_{11}^4 - \bar{h}_{12}^3 \bar{h}_{12}^4 = 0, \tag{35}$$

which is independent of the choice of  $\{e_i\}$ .

Set

$$\hat{e}_1 = e_1 \cosh \theta + e_2 \sinh \theta, \quad \hat{e}_2 = e_1 \sinh \theta + e_2 \cosh \theta$$

for a smooth function  $\theta$ , and let  $\{\hat{h}_{ij}^\alpha\}$  be the components of  $h$  with respect to the frame  $\{\hat{e}_i, \bar{e}_\alpha\}$ . Then we have

$$\begin{aligned}\hat{h}_{11}^3 &= \bar{h}_{11}^3 \cosh 2\theta + \bar{h}_{12}^3 \sinh 2\theta, & \hat{h}_{12}^3 &= \bar{h}_{11}^3 \sinh 2\theta + \bar{h}_{12}^3 \cosh 2\theta, \\ \hat{h}_{11}^4 &= \bar{h}_{11}^4 \cosh 2\theta + \bar{h}_{12}^4 \sinh 2\theta, & \hat{h}_{12}^4 &= \bar{h}_{11}^4 \sinh 2\theta + \bar{h}_{12}^4 \cosh 2\theta.\end{aligned} \tag{36}$$

As we assume that  $K > c$ , we have by (10),

$$(\bar{h}_{11}^4)^2 - (\bar{h}_{12}^4)^2 > (\bar{h}_{11}^3)^2 - (\bar{h}_{12}^3)^2.$$

So  $(\bar{h}_{11}^4)^2 - (\bar{h}_{12}^4)^2 > 0$ , or  $(\bar{h}_{11}^3)^2 - (\bar{h}_{12}^3)^2 < 0$ . When  $(\bar{h}_{11}^4)^2 - (\bar{h}_{12}^4)^2 > 0$ , we may choose the smooth function  $\theta$  so that  $\hat{h}_{12}^4 = 0$ . Then  $\hat{h}_{11}^4 \neq 0$ , and by (35),  $\hat{h}_{11}^3 = 0$ . Similarly, when  $(\bar{h}_{11}^3)^2 - (\bar{h}_{12}^3)^2 < 0$ , we may choose the smooth function  $\theta$  so that  $\hat{h}_{11}^3 = 0$ . Then  $\hat{h}_{12}^3 \neq 0$ , and by (35),  $\hat{h}_{12}^4 = 0$ .

Thus, with respect to the frame  $\{\hat{e}_i, \hat{e}_\alpha\}$ , we have

$$\hat{\omega}_1^3 = t\hat{\omega}^2, \quad \hat{\omega}_2^3 = t\hat{\omega}^1, \quad \hat{\omega}_1^4 = u\hat{\omega}^1, \quad \hat{\omega}_2^4 = u\hat{\omega}^2,$$

and

$$K - c = t^2 + u^2, \quad K_\nu = -2tu.$$

Let  $q$  and  $r$  be defined as in (33) and (34). Then we have  $(t, u) = \pm(q, r)$ , or  $(t, u) = \pm(r, q)$ .

Hence we get the following:

**LEMMA 5.1.** *Let  $M$  be a Lorentzian stationary surface in  $N_2^4(c)$  satisfying  $(K - c)^2 - K_\nu^2 > 0$  and  $K > c$ . Then we may choose the frame  $\{e_A\}$  so that*

$$\omega_1^3 = q\omega^2, \quad \omega_2^3 = q\omega^1, \quad \omega_1^4 = r\omega^1, \quad \omega_2^4 = r\omega^2,$$

or

$$\omega_1^3 = r\omega^2, \quad \omega_2^3 = r\omega^1, \quad \omega_1^4 = q\omega^1, \quad \omega_2^4 = q\omega^2.$$

**PROOF OF THEOREM 1.2.** (ii) We may assume that  $K > c$ . For the Lorentzian stationary immersion  $f$ , we choose the frame  $\{e_A\}$  as in Lemma 5.1. Let  $\hat{f}: M \rightarrow N_2^4(c)$  be an arbitrary isometric stationary immersion with the same normal curvature  $K_\nu$ . By Lemma 5.1, we may choose the frame  $\{\hat{e}_A\}$  so that

$$\hat{\omega}_1^3 = q\hat{\omega}^2, \quad \hat{\omega}_2^3 = q\hat{\omega}^1, \quad \hat{\omega}_1^4 = r\hat{\omega}^1, \quad \hat{\omega}_2^4 = r\hat{\omega}^2,$$

or

$$\hat{\omega}_1^3 = r\hat{\omega}^2, \quad \hat{\omega}_2^3 = r\hat{\omega}^1, \quad \hat{\omega}_1^4 = q\hat{\omega}^1, \quad \hat{\omega}_2^4 = q\hat{\omega}^2.$$

Then, as in (28) and (29), we have  $\hat{\omega}_2^1 = \omega_2^1$  and  $\hat{\omega}_4^3 = \omega_4^3$ .

Also as in (32), there exist coordinate systems  $\{x^1, x^2\}$  and  $\{\hat{x}^1, \hat{x}^2\}$  such that

$$\omega^i = \{(K - c)^2 - K_\nu^2\}^{-1/8} dx^i,$$

and

$$\hat{\omega}^i = \{(K - c)^2 - K_\nu^2\}^{-1/8} d\hat{x}^i.$$

We may write

$$\frac{\partial}{\partial \hat{x}^1} = \cosh \theta \frac{\partial}{\partial x^1} - \sinh \theta \frac{\partial}{\partial x^2}, \quad \frac{\partial}{\partial \hat{x}^2} = -\sinh \theta \frac{\partial}{\partial x^1} + \cosh \theta \frac{\partial}{\partial x^2}$$

for a smooth function  $\theta$ . As  $[\partial/\partial \hat{x}^1, \partial/\partial \hat{x}^2] = 0$ , we find that  $\theta$  is constant. We note that

$$e_1 = (\cosh \theta) \hat{e}_1 + (\sinh \theta) \hat{e}_2, \quad e_2 = (\sinh \theta) \hat{e}_1 + (\cosh \theta) \hat{e}_2.$$

Using (36) in this situation, we can see that the components of the second fundamental form of  $\hat{f}$  with respect to the frame  $\{e_i, \hat{e}_\alpha\}$  are the same as those of  $f_\theta$  or  $\bar{f}_\theta$  with respect to  $\{e_i, e_\alpha\}$ . Also, with respect to those frames,  $\hat{\omega}_4^3 = \omega_4^3 = \omega_4^3(\theta) = \bar{\omega}_4^3(\theta)$ , that is,  $\hat{f}$ ,  $f_\theta$  and  $\bar{f}_\theta$  have the same normal connection. Therefore,  $\hat{f}$  coincides with  $f_\theta$  or  $\bar{f}_\theta$  up to congruence.  $\square$

REMARK. In the case where  $(K - c)^2 - K_v^2 \leq 0$ , we may not choose the frame so that the equation (35) is satisfied. That is,  $(T^{\alpha\beta})$  given by

$$T^{\alpha\beta} = h_{11}^\alpha h_{11}^\beta - h_{12}^\alpha h_{12}^\beta$$

may not be diagonalized. It should be a crucial different point compared with the case where  $(K - c)^2 - K_v^2 > 0$ .

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